Axler Ch 1A - 1B

→ Partition of $[a,b]$ $\quad P = \{x_0 \cdots x_n\} \quad a = x_0 < \cdots < x_n = b$

→ Lower sum: $\quad$ function $f : [a,b] \longrightarrow \mathbb{R}$

will drop this 3rd argument b'cos we're lazy.

$$L(f, P, [a,b]) \quad \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{[x_{i+1}, x_i]} f$$

$$U(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{[x_{i+1}, x_i]} f$$

Ex: $\quad f = x^2$ on $[0,1]$ $\quad P_n = 0 < \frac{1}{n} < \frac{2}{n} \cdots < 1$

$$L(x^2, P_n) = \sum_{i=0}^{n-1} \frac{1}{n} \cdot \frac{i^2}{n^2} = \frac{1}{n^3}\left[\frac{(n-1)(n)(2n-1)}{6}\right] = \frac{1}{6n^2}\left[2n^2 - 3n + 1\right]$$

$$U(x^2, P_n) = \sum_{i=1}^{n} \frac{1}{n} \frac{i^2}{n^2} = \frac{1}{6n^2}\left[2n^2 + 3n + 1\right]$$

Prop: $\quad L(f, P) \leq U(f, P)$

Refinements: $\quad P'$ is a refinement of P if $\quad P \subset P'$

Prop: $\quad L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P)$

Wonderful property: tells you how U, L are being squeezed toward each other as P is refined

How about two arbitrary P, P'?

Can you say

Prop 1.6 $\quad L(f, P) \leq U(f, P')$ YES!

Consider $P'' = P \vee P'$ (its just union)

Lower & Upper Integrals: $L(f, [a,b]) = \sup_P L(f, P, [a,b])$

$$U(f, [a,b]) = \inf_P U(f, P, [a,b])$$

Defn: (Riemann Integrability): If f bounded on $[a,b]$, then its Riemann integrable if

$$L(f, [a,b]) = U(f, [a,b])$$

Ex: $\lim_{n\to\infty} L(f, P_n) \leq L(f, [a,b]) \leq U(f, [a,b]) \leq \frac{1}{3}$

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} \frac{1}{6n^2}\left[2n^2 - 3n + 1\right] = \frac{1}{3}$$

Theorem: If $f : [a,b] \to \mathbb{R}$ is continuous, then its Riemann.

Pf: Uniform continuity (follows from compactness)

Let $\varepsilon > 0$, $\exists\ \delta$ st $|f(s) - f(t)| < \epsilon \quad \forall\ |t-s| < \delta$, $s,t \in [a,b]$

Choose a fine partition $P_n = \left\{a, a + \frac{b-a}{n}, \dots, b - \frac{b-a}{n}, b\right\} = \left\{x_0, \dots, x_n\right\}$

For $\frac{1}{n} < \delta$

$$L(f, P_n) = \sum_{i=0}^{n-1} (x_i - x_{i-1}) \inf_{[x_{i-1}, x_i]} f \geq \sum_{i=0}^{n-1} (x_i - x_{i-1})\left(\sup_{[x_{i-1}, x_i]} f - \epsilon\right)$$

$$= U(f, P_n) - \epsilon$$

ϵ is arbitrary, so done.

1B : Riemann integral is not good enough.

Ex: $f = \begin{cases} 1 & x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$

$U(f,P) = 1 \qquad L(f,P) = 0 \qquad \forall$ partn P.

"Disturbing example".

- Does not work with unbounded functions: $f(x) = \frac{1}{\sqrt{x}}$ on $[0,1]$

Because $[0,1]$, any partn will contain $[0,a]$ as an interval.

The upper sum will be unbounded.

- How to fix? $\lim_{a \downarrow 0} \int_a^1 f(x)\,dx = (2 - 2\sqrt{a})$

★ Exercise: using just the tools we have built so far.

Ex: Let $\mathbb{Q} \cap [0,1] = \{r_0, r_1, \dots\}$ be an enumeration.

$f_n = \begin{cases} \frac{1}{\sqrt{x - r_n}} & x > r_n \\ 0 & x \le r_n \end{cases}$ From previous $\int_{r_n}^1 f_n = 2\sqrt{x}\Big|_0^{1-r_n} = 2\sqrt{1-r_n}$

$f = \sum_{n=0}^{\infty} \frac{1}{2^n} f_n \qquad \int f \le 2 \sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \cdot \frac{1}{1 - 1/2} = 4$

So it ought to be integrable. But you cannot break it up into integrals over some intervals since it will always contain an r_n & f is unbounded at r_n.

Q★: Can you compute the value of $\int f$?

Bad under pointwise limits

Let $f_n = \begin{cases} 1 & x \in \{r_1, \dots, r_n\} \\ 0 & \end{cases}$

f_n all Riemann integrable. It's an exercise on your HW.

$f_k(x) \longrightarrow f(x)$, but f_n is not integrable.

<u>Bounded Convergence Theorem</u>

Suppose $\{f_i\}_{i=1}^{\infty}$ are Riemann integrable & uniformly bounded. Suppose

$$\lim_{n\to\infty} f_n = f(x) \quad \forall x \in [a,b]$$

$\boxed{\text{If}}$ f is Riemann integrable then $\lim_{n\to\infty} \int f_n = \int f$

☆ Cf previous example.

☆ Can you prove this? P. Deift once told me that this was where one realizes the need for the Riemann integrable.

- We will prove a theorem in the exercises that says if two Riemann integrable fns agree on all but a FINITE set of points then their integrals are equal.
- They key here is FINITE. When we move to Lebesgue measure we will upgrade this to COUNTABLE. (Look up Jordan measure)

1B 1,3.